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Stochastic Realization of a Gaussian Stochastic Control System*

J. H. VAN SCHUPPEN
CWI – Centre for Mathematics and Computer Science, PO Box 94079, 1090 GB Amsterdam, The Netherlands

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Abstract. The stochastic realization problem is considered of representing a stationary Gaussian process as the observation process of a Gaussian stochastic control system. The problem formulation includes that the last m components of the observation process form the Gaussian white noise input process to the system. Identifiability of this class of systems motivates the problem. The results include a necessary and sufficient condition for the existence of a stochastic realization. A subclass of Gaussian stochastic control systems is defined that is almost a canonical form for this stochastic realization problem. For a structured Gaussian stochastic control system an equivalent condition for identifiability of the parametrization is stated.

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1. Introduction

The purpose of this paper is to present the problem and the solution to a stochastic realization problem for a Gaussian stochastic control system.

The motivation for the stochastic realization problem is system identification. The research area of system identification addresses the problem of how to obtain a mathematical model for an observed phenomenon. A major question in system identification is how to select a parametrization of the model class.

In this paper, attention is restricted to the class of time-invariant Gaussian stochastic control systems. An element of this class is represented by the relations

$$x(t+1) = Ax(t) + Bu(t) + Mv(t),$$
$$y(t) = Cx(t) + Du(t) + Nv(t),$$

in which u represents the input process, v a Gaussian white noise process, x the state process, and y the output process. See Section 2 for a formal definition. Motivated

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by the practice of system identification, the input process is assumed to be Gaussian white noise. The observation process consists then of the input and output process combined which leads to the representation

$$x(t+1) = Ax(t) + \begin{pmatrix} M & B \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix},$$

$$z(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} N & D \\ 0 & I \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}.$$

The problem is then to select a parametrization for the class of Gaussian stochastic control systems and to determine whether this parametrization is identifiable or not.

In the literature identifiability conditions for this class of systems are stated at several places. L. Ljung mentions such conditions for systems in ARMAX representation [14, pp. 101–106]. However, it is not clear from this description how these conditions are derived. The case of a state space representation is restricted to a single-input-single-output system. In the book by E. J. Hannan and M. Deistler [8], conditions are stated under which a Gaussian stochastic control system is minimal, [8, Theorem 2.3.3]. However, the starting point in this book is a spectral factor while in the author's opinion, it should be the covariance function or the spectral density. However, within the selected framework, the book [8] presents a coherent theory.

It seems to the author of this paper that conditions for identifiability of a class of dynamic systems should be based on stochastic realization theory. Questions as to the minimality of a dynamic system and the equivalence class of minimal realizations must first be studied within a theoretical framework. The answers to these questions will then provide conditions for identifiability of a dynamic system. This approach has not been pursued for the class of Gaussian stochastic control systems. There is also a need for identifiability of structured Gaussian stochastic control systems. In many applications, the system is based on physical laws. This results in a representation that is structured, the matrices of the system contain as elements zeroes and free parameters. Identifiability conditions for such structured systems are currently not available.

The problem of this paper is to show that a stationary Gaussian process can be represented as the observation process of a Gaussian stochastic control system. The problem specification includes that the last m components of this observation process form a Gaussian white noise process that is to be considered as input process. If such a system exists, then it will be called a (weak) stochastic realization of the given process. Attention is restricted to those stochastic realizations of which the state space is of minimal dimension. In general, many minimal stochastic realizations exist. Of interest, then, is a classification of the class of minimal stochastic realizations and a canonical form specifying one element in each equivalence class. Stochastic realization theory is the name for the research area in which these problems are studied. Identifiability conditions follow immediately from stochastic realization theory.

A summary of the paper by section follows. In Section 2, definitions are presented and the problem formulated. The existence of a stochastic realization is proven in Section 3 and an algorithm is provided. The issues of a parametrization and a canonical form are solved in Section 4.

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2. Problem Formulation

In this section, notation is introduced and the problem defined.

System identification is a research area that addresses the problem of how to construct from data a mathematical model in the form of a dynamic system that is realistic and not overly complex. System identification is practised in engineering, econometrics, statistics, and biology.

A procedure for system identification of a phenomenon may consist of the following steps:

(1) Modeling of the phenomenon using prior knowledge of the domain. Selection of a model class of dynamic systems.
(2) Experimentation and data collection.
(3) Parametrization of the model class and a check on the identifiability of the parametrization.
(4) Selection of a model in the model class. This may be done by a least-squares algorithm or a maximum likelihood algorithm.
(5) Evaluation of the selected model. Possibly repetition of one or more of the above steps.

In this paper, attention is restricted to the model class of time-invariant Gaussian stochastic control systems. The notation used is fairly standard. Let $C^- = \{c \in C \mid |c| < 1\}$. If $A \in R^{n \times n}$, then $\text{sp}(A)$ denotes the spectrum of A.

From the geometric approach to linear systems, the notation is

$$\langle A \mid \text{Im}(B) \rangle = \{Sx \in R^n \mid \forall x \in R^n\},$$

where

$$S = (B \quad AB \quad \dots \quad A^{n-1}B).$$

The real line is equipped with the Borel σ-algebra, denoted by B. That of R^n is denoted by B^n. If $x\colon \Omega \to R^n$ is a random variable with a Gaussian distribution having mean value $m \in R^n$ and variance $Q \in R^{n \times n}$, then this is denoted by $x \in G(m, Q)$. The σ-algebra generated by x is denoted by F^x and the σ-algebra family of a stochastic process $v\colon \Omega \times T \to R^r$ by $\{F_t^v,\ t \in T\}$.

DEFINITION 2.1. A time-invariant Gaussian stochastic control system is a collection

$$\{\Omega, F, P, T, R^p, B^p, R^n, B^n, R^m, B^m, y, x, u\}$$

in which the state process x and the output process y are determined by the recursions

$$x(t+1) = Ax(t) + Bu(t) + Mv(t), \qquad x(t_0) = x_0, \tag{1}$$

$$y(t) = Cx(t) + Du(t) + Nv(t), \tag{2}$$

where (Ω, F, P) is a probability space, $T = \{t_0, t_0 + 1, \ldots\} \subset Z$ is a time index set,

$$X = R^n, \quad B_X = B^n, \quad Y = R^p, \quad B_Y = B^p,$$

$$U = R^m, \quad B_U = B^m,$$

$$x_0\colon \Omega \longrightarrow X, \quad x_0 \in G(m_0, Q_0),$$

$v\colon \Omega \times T \to R^r$ is a Gaussian white noise process with $v(t) \in G(0, V)$, $V = V^T > 0$, $u\colon \Omega \times T \to U$ is a stochastic process with F^{x_0}, F^v_∞, and F^u_∞ independent σ-algebras, and $x\colon \Omega \times T \to X$ and $y\colon \Omega \times T \to Y$ defined by (1), (2). If $T = Z$, as is the case in the rest of the paper, then the state at $t = -\infty$ is supposed to be an equilibrium state. These equations are referred to as the Gaussian stochastic control system representation of the corresponding system. The parameters of a Gaussian stochastic control system are denoted by

$$\{A, B, C, D, M, N, V\} \in GSCSP(p, n, m, r), \tag{3}$$

for $p, n, m, r \in Z_+$.

The second step in the system identification procedure is experimentation and data collection. Experimentation of a stochastic control system often proceeds by supplying a pseudo-random time series as input and collecting the resulting input and output time series. In practice, a pseudo-random two-valued time series is used. The pseudo-random two-valued input time series may be modelled as a trajectory of a Gaussian white noise process. This modeling approximation is of minor importance.

ASSUMPTION 2.2. The input process of the Gaussian stochastic control system considered is a stationary Gaussian white noise process with $u(t) \in G(0, V_u)$, $V_u = V_u^T > 0$.

This assumption is imposed throughout the paper.

The third step of the system identification procedure is to select a parametrization of the model class and to check whether this parametrization is identifiable. Conditions for identifiability of a dynamic system may be based on realization theory as developed in systems theory. Below, the stochastic realization problem is posed for the class of Gaussian stochastic control systems.

PROBLEM 2.3. The weak stochastic realization problem for a stationary Gaussian stochastic control system. Assume given a stationary Gaussian process on $T = Z$ taking values in (R^{p+m}, B^{p+m}) having mean value function zero and covariance function $W: T \to R^{(p+m)\times(p+m)}$. Assume that the last $m \geqslant 1$ components of this process are the input process and Gaussian white noise with a nonsingular variance. Solve the following subproblems.

(a) Does there exist a time-invariant Gaussian stochastic control system such that the observation process of outputs and inputs equals the given process in distribution? Or, if the system is given by

$$x(t+1) = Ax(t) + \begin{pmatrix} M & B \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}, \tag{4}$$

$$z(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} N & D \\ 0 & I \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}, \tag{5}$$

with u a Gaussian white noise process $u(t) \in G(0, V_u)$ independent of the process v, is then the observation process z equal to the given process in distribution? Equality in distribution means that the family of finite-dimensional distributions of both processes are identical. Because both processes are stationary Gaussian with mean value function equal to zero, the equality in distribution is equivalent to equality of the corresponding covariance functions.

If such a system exists, then one calls it a *weak stochastic realization* of the given process, or, if the context is known, a *stochastic realization*.

(b) Classify all minimal stochastic realizations of the given process. A weak stochastic realization is called *minimal* if the dimension of the state space is minimal. The following points must be addressed:

(1) characterize those stochastic realizations that are minimal;
(2) obtain the classification as such;
(3) indicate the relation between two minimal stochastic realizations;
(4) produce an algorithm that constructs all minimal weak stochastic realizations of the given process.

It is argued that Problem 2.3 is relevant for system identification of Gaussian stochastic control systems. Consider the least-squares estimation or maximum likelihood estimation of the parameters of such a control system. As may be seen from the book [14, pp. 176–177], estimation algorithms for these two criteria are equivalent to selection of an element in the model class that minimize the distance between two covariance functions. The first covariance function is that of the considered system in the model class and the second is that of the estimated covariance function according to, for example, the formula's

$$\bar{y} = \frac{1}{t_1} \sum_{s=1}^{t_1} y(s),$$

$$\widehat{W}_z(t) = \frac{1}{t_1} \sum_{s=1}^{t_1} \begin{pmatrix} y(s+t) - \bar{y} \\ u(s+t) \end{pmatrix} \begin{pmatrix} y(s) - \bar{y} \\ u(s) \end{pmatrix}^T .$$

The fitting of this estimated covariance function may now be abstracted to the stochastic realization Problem 2.3. In this approach, the given process is specified by its family of finite-dimensional distributions and because it is a Gaussian process with mean value zero, it is mainly specified by its covariance function.

The stochastic realization Problem 2.3 is a special case of one that has been formulated by the author in 1985, see [17, Problem 4.2].

The stochastic realization problem in which the input process does depend on the output process is not discussed in this paper. See [1, 2] for references on this problem. The stochastic realization problem in which it is not known which components of the observation process belong to the input and which to the output, is not discussed here. In the literature of econometrics there are papers on the causality relation between stochastic processes, see [7] for an entry to the literature.

3. Existence of a Stochastic Realization and an Algorithm

In this section, the existence part of the stochastic realization problem for a Gaussian stochastic control system will be solved and an algorithm stated.

The stochastic realization problem for a time-invariant Gaussian stochastic system without input process has a long history. For references, see [5, 6]. For references on the strong stochastic realization problem for Gaussian systems, see [11–13].

THEOREM 3.1. *Consider the stochastic realization Problem 2.3. Assume that the covariance function satisfies* $\lim_{t\to\infty} W(t) = 0$ *and* $W(0) > 0$. *Assume that the last* m *components of the observation process are inputs and Gaussian white noise.*

(a) *There exists a Gaussian stochastic control system that is a weak stochastic realization of the given process iff*

(1) *the infinite Hankel matrix associated with the covariance function has finite rank* (*see* [4] *or* [15] *for the definition of a Hankel matrix associated with an impuls response function*);

(2) $W_{21}\colon T \to R^{m\times p}$, $W_{21}(t) = 0,\ t > 0,$ (6)
where W_{21} *is the* $m \times p$ *subblock of* W.

Denote a stochastic realization by

$$x(t+1) = Ax(t) + \begin{pmatrix} M & B \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix},$$

$$z(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} N & D \\ 0 & I \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}.$$

(b) *A stochastic realization as defined in* (a) *is minimal iff*

(1) *the state space is the support of the state process, or, equivalently, if* $(A, [MV \mid BV_u])$ *is a reachable pair;*

(2) *the stochastic system is stochastically observable, or* (A, C) *is an observable pair;*

(3) *the stochastic system is stochastically reconstructible, or* $(A, [G_1 \mid B])$ *is a reachable pair where*

$$G_1 = AQC^T + MVN^T, \tag{7}$$

$$Q = AQA^T + MVM^T + BV_u B^T, \tag{8}$$

and where $Q \in R^{n \times n}$ *is the unique solution of the discrete Lyapunov equation* (8).

(c) *Assume that the covariance function satisfies the conditions of* (a). *Then the class of minimal stochastic realizations is described by:*

(1) *one particular system and a state space isomorphism as in finite-dimensional linear systems or in Gaussian systems without input;*

(2) *for any fixed isomorphism, the class is in bijective correspondence with the set*

$$\mathrm{Q}_1 = \{Q \in R^{n \times n} \mid Q = Q^T > 0,\ V(Q) \geqslant 0\}, \tag{9}$$

$$V(Q) = \begin{pmatrix} Q - FQF^T & G - FQH^T \\ G^T - HQF^T & 2J - HQH^T \end{pmatrix} \in R^{(p+n+m)\times(p+n+m)}. \tag{10}$$

(d) *The stochastic realization Algorithm* 3.2 *given below is well defined and determines a stochastic realization in the form of a Gaussian stochastic control system.*

ALGORITHM 3.2. Calculation of a weak stochastic realization of a Gaussian stochastic control system. Data: Covariance function $W\colon T \to R^{(p+m)\times(p+m)}$ and dimension $m \in Z_+$ with $0 < m < p + m$ of the input process.

(1) If the infinite Hankel matrix associated with the covariance function is finite, say with rank $n \in Z_+$, then determine $F \in R^{n\times n}$, $G \in R^{(p+m)\times n}$, $H \in R^{n\times(p+m)}$, $J \in R^{(p+m)\times(p+m)}$ such that

$$W(t) = \begin{cases} 2J, & t = 0, \\ HF^{t-1}G, & t > 0, \end{cases} \tag{11}$$

by a well known realization algorithm for finite-dimensional linear systems; (see for algorithms [4] or [15]); else stop;

(2) Determine $Q \in \mathrm{Q}_1$ and $V(Q)$. See [6] for algorithms.

(3) Let

$$A = F, \qquad C = H, \qquad M = (I_n \quad 0), \qquad N = (0 \quad I_p),$$
$$V = V(Q), \quad w(t) \in G(0, V), \tag{12}$$

$$x(t+1) = Ax(t) + Mw(t), \tag{13}$$

$$z(t) = Cx(t) + Nw(t), \tag{14}$$

(4) Partition the observation process into outputs and inputs according to

$$z(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \qquad N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \tag{15}$$

$C_1 \in R^{p \times n}$, $C_2 \in R^{m \times n}$, $N_1 \in R^{p \times k}$, $N_2 \in R^{m \times k}$. Then $C_2 = 0$.

(5) Compute $T_1^{(p+n+m) \times (p+n+m)}$ such that

$$V = T_1 T_1^T \tag{16}$$

and $S \in R^{k \times k}$ such that $SS^T = I$ and

$$N_2 T_1 S = (0 \quad L) \tag{17}$$

with $L \in R^{m \times m}$. Let

$$\begin{pmatrix} M_1 & M_2 \\ N_{11} & N_{12} \\ 0 & L \end{pmatrix} = \begin{pmatrix} M \\ N_1 \\ N_2 \end{pmatrix} T_1 S, \tag{18}$$

with $M_1 \in R^{n \times (p+n)}$, $M_2 \in R^{n \times m}$, $N_{11} \in R^{p \times (p+n)}$, $N_{12} \in R^{p \times m}$. Let

$$\begin{pmatrix} v(t) \\ u(t) \end{pmatrix} = S^T T_1^{-1} w(t). \tag{19}$$

(6) The stochastic realization is then

$$x(t+1) = Ax(t) + (M_1 \quad M_2) \begin{pmatrix} v(t) \\ u(t) \end{pmatrix},$$

$$z(t) = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} N_{11} & N_{12} \\ 0 & L \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix},$$

where u, v are standard independent Gaussian white noise processes

The above algorithm is not completely determined, it involves two choices, one in each of the steps 1 and 2.

Before presenting the proof of Theorem 3.1, examples are presented that illustrate the minimality condition.

EXAMPLE 3.3. Consider the Gaussian stochastic control system with $p = 2$, $n = 3$, $m = 1$, and $r = 2$,

$$x(t+1) = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 & \frac{1}{2}\sqrt{3} \\ \frac{2}{3}\sqrt{2} & 0 & 0 \\ 0 & \frac{1}{4}\sqrt{15} & 0 \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}, \tag{20}$$

$$y(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}, \tag{21}$$

with $v(t) \in G(0, I)$, $u(t) \in G(0, 1)$. This system is a minimal stochastic realization of its observation process.

EXAMPLE 3.4. Consider the Gaussian stochastic control system with $p = 2$, $n = 3$, $m = 1$, and $r = 2$,

$$x(t+1) = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & -3/2 & 0 \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}, \tag{22}$$

$$y(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}, \tag{23}$$

with $v(t) \in G(0, I)$, $u(t) \in G(0, 1)$. This system is not a minimal stochastic realization of its observation process. The process y_2 is Gaussian white noise. This system is a minimal stochastic realization according to [8, Theorem 2.3.3].

EXAMPLE 3.5. Consider the Gaussian stochastic control system with $p = 2$, $n = 3$, $m = 1$, and $r = 2$,

$$x(t+1) = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} x(t) + \begin{pmatrix} \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & \frac{2}{3}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{4}\sqrt{15} \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}, \tag{24}$$

$$y(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} x(t) + \begin{pmatrix} \frac{2}{3}\sqrt{3} & 0 & \frac{1}{15}\sqrt{15} \\ 0 & -\frac{1}{4}\sqrt{2} & \frac{2}{15}\sqrt{15} \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}, \tag{25}$$

with $v(t) \in G(0, I)$, $u(t) \in G(0, 1)$. This system is not a minimal stochastic realization of its observation process. The conditions for minimality are:

(1) $(A, [M \mid B])$ is a reachable pair, which condition is satisfied;
(2) (A, C) is an observable pair, which condition is satisfied;
(3) $(A, [G_1 \mid B])$ is a reachable pair, which condition is not satisfied, because

$$\langle A \mid \mathrm{Im}(G_1)\rangle \cup \langle A \mid \mathrm{Im}(B)\rangle = \mathrm{span} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \neq R^3 = X.$$

The nonminimality is illustrated by the backward representation of the stochastic system

$$x(t-1) = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} x(t) + \begin{pmatrix} \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & \frac{2}{3}\sqrt{2} & 0 \\ 0 & 0 & \frac{1}{4}\sqrt{15} \end{pmatrix} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}, \tag{26}$$

$$y(t) = \begin{pmatrix} 3/2 & 0 & 1/2 \\ 0 & 0 & 3/4 \end{pmatrix} x(t) + \begin{pmatrix} N & D \end{pmatrix} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}, \tag{27}$$

This system is a minimal stochastic realization according to [14, p. 105].

The reason why minimality does not hold in Examples 3.4 and 3.5 is because condition 3 of Theorem 3.1.b, stochastic reconstructibility, is not satisfied. In [8, p. 37], two ARMAX systems are said to be observationally equivalent if the two transfer matrices coincide. These matrices correspond to a spectral factor. In this paper, the spectral density is taken as the starting point because in system identification the estimated covariance function is the basic object. The same comment applies to the approach of [14]. In [14, p. 105], minimality conditions are presented for a state space realization. The conditions for minimality differ from those given here.

There are several articles that discuss stochastic controllability of Gaussian stochastic control systems. None of the definitions given in these papers seems useful in the context of Theorem 3.1.

Below follows the proof of Theorem 3.1.

PROPOSITION 3.6. *Consider a Gaussian stochastic control system in the representation* (4), (5). *Assume that* $\mathrm{sp}(A) \subset C^-$. *Asymptotically, for large times, the observation process* z *of this system is a stationary Gaussian process with zero mean value function and covariance function*

$$W_z(t) = \begin{cases} \begin{pmatrix} CA^{t-1}G & CA^{t-1}BV_u \\ 0 & 0 \end{pmatrix}, & \text{if } t > 0, \\ \begin{pmatrix} CQC^T + NVN^T + DV_uD^T & DV_u \\ V_uD^T & V_u \end{pmatrix}, & \text{if } t = 0, \end{cases} \tag{28}$$

where $Q \in R^{n \times n}$ *is the unique solution of the Lyapunov equation*

$$Q = AQA^T + MVM^T + BV_uB^T, \tag{29}$$

and

$$G = AQC^T + MVN^T + BV_uD^T. \tag{30}$$

Proof. This is a direct calculation using results for stochastic realization theory of Gaussian stochastic control systems without input. □

Proof of Theorem 3.1 (1) By stochastic realization theory for Gaussian stochastic systems, the observation process has a stochastic realization as a Gaussian system representation

$$x(t+1) = Ax(t) + Mw(t), \tag{31}$$

$$z(t) = Cx(t) + Nw(t), \tag{32}$$

where $w: \Omega \times T \rightarrow R^k$ is a stationary Gaussian white noise process, say with $w(t) \in G(0, V_w)$, $V_w = V_w^T > 0$. Suppose that a minimal stochastic realization has been selected.

(2) It is given that the last m components of the observation process are Gaussian white noise. Partition z as

$$z(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix},$$

where $y: \Omega \times T \rightarrow R^p$ and $u: \Omega \times T \rightarrow R^m$. Then u is a stationary Gaussian white noise, say $u(t) \in G(0, V_u)$. Partition conform the partition of z

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \qquad N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix},$$

where $C_1 \in R^{p\times n}$, $C_2 \in R^{m\times n}$, $N_1 \in R^{p\times k}$, $N_2 \in R^{m\times k}$.

(3) From the fact that the system (31, 32) is a stochastic realization it follows that

$$Q = AQA^T + MV_wM^T, \tag{33}$$

$$G = AQC^T + MV_wN^T, \tag{34}$$

$$\begin{pmatrix} W_{uy}(t) & W_{uu}(t) \end{pmatrix} = \begin{cases} C_2QC_2^T + N_2V_wN_2^T, & t = 0, \\ C_2A^{t-1}G, & t > 0, \end{cases} \tag{35}$$

$$W_z(t) = CA^{t-1}G, \quad t > 0. \tag{36}$$

Because the stochastic realization is by construction minimal it follows that (A, G) is a reachable pair or

$$\text{rank}\begin{pmatrix} G & AG & \dots & A^{n-1}G \end{pmatrix} = n.$$

By assumption

$$\begin{pmatrix} W_{uy}(t) & W_{uu}(t) \end{pmatrix} = 0, \quad \text{for } t > 0.$$

Hence

$$0 = W_{u.}(t) = C_2A^{t-1}G, \quad \forall t > 0,$$

implies that $C_2 = 0$.

(4) Let $T_1 \in R^{k\times k}$ be such that $V_w = T_1 T_1^T$. Because V_w is nonsingular, $\mathrm{rank}(T_1) = k$. Consider $N_2 T_1 \in R^{m\times k}$. Note that $m \leqslant k$. Then there exists a $S \in R^{k\times k}$ such that $SS^T = I$, or S is orthogonal, and

$$N_2 T_1 S = (0 \quad L),$$

where $L \in R^{m\times m}$. This follows from the singular value decomposition of $N_2 T_1$ according to

$$N_2 T_1 = S_1 (0 \quad L_2) S_2^T = (0 \quad S_1 L_2) S_2^T.$$

Define

$$\begin{pmatrix} M_1 & M_2 \\ N_{11} & N_{12} \\ 0 & L \end{pmatrix} = \begin{pmatrix} M \\ N_1 \\ N_2 \end{pmatrix} T_1 S,$$

and let $v\colon \Omega \times T \to R^{p+n}$, $u\colon \Omega \times T \to R^m$

$$\begin{pmatrix} v(t) \\ u(t) \end{pmatrix} = S^T T_1^{-1} w(t).$$

Then this process is Gaussian white noise with variance

$$S^T T_1^{-1} V_w T_1^{-T} S^{-T} = S^T T_1^{-1} T_1 T_1^T T_1^{-T} S^{-T} = I,$$

hence v, u are independent standard Gaussian white noise processes. Then

$$\begin{pmatrix} M \\ N_1 \\ N_2 \end{pmatrix} w(t) = \begin{pmatrix} M \\ N_1 \\ N_2 \end{pmatrix} T_1 S S^T T_1^{-1} w(t) = \begin{pmatrix} M_1 & M_2 \\ N_{11} & N_{12} \\ 0 & L \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}.$$

Note that

$$W_z(0) = \begin{pmatrix} C_1 Q C_1^T + M_1 M_1^T + M_2 M_2^T & M_1 N_{11}^T + M_2 N_{12}^T & M_2 L^T \\ N_{11} M_1^T + N_{12} M_2^T & N_{11} N_{11}^T + N_{12} N_{12}^T & N_{12} L^T \\ L M_2^T & L N_{12}^T & L L^T \end{pmatrix}.$$

Then the assumption $W_z(0) > 0$ implies that $\mathrm{rank}(LL^T) = m$ or $\mathrm{rank}(L) = m$.

(5) The stochastic realization is then

$$x(t+1) = Ax(t) + (M_1 \quad M_2) \begin{pmatrix} v(t) \\ u(t) \end{pmatrix},$$

$$z(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} N_{11} & N_{12} \\ 0 & L \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix},$$

in which v, u are independent standard Gaussian white noise processes.

(6) From stochastic realization theory for a Gaussian stochastic system without input follows that the system (31), (32) is minimal iff

(1) the state space is the support of the state process;
(2) the system is stochastically observable;
(3) the system is stochastically reconstructible.

With $x(t) \in G(0, Q)$, condition (1) is equivalent to $Q > 0$. Now $\mathrm{sp}(A) \subset C^-$ and

$$\begin{aligned} Q &= AQA^T + MVM^T + BV_u B \\ &= AQA^T + \begin{pmatrix} MV^{1/2} & BV_u^{1/2} \end{pmatrix} \begin{pmatrix} MV^{1/2} & BV_u^{1/2} \end{pmatrix}^T . \end{aligned}$$

From a well-known result on the Lyapunov equation it follows that $Q > 0$ iff $(A, [MV^{1/2} \mid BV_u^{1/2}])$ is a reachable pair.

From stochastic realization of a Gaussian stochastic system without input and condition (1) follows that the system is stochastically observable iff (A, C) is an observable pair. Similarly, it follows that the system is stochastically reconstructible iff (A, G) is a reachable pair in which

$$G = \begin{pmatrix} AQC^T + MVN^T + BV_u D^T & BV_u \end{pmatrix} = \begin{pmatrix} G_1 + BV_u D^T & BV_u \end{pmatrix}.$$

Now (A, G) is a reachable pair iff

$$\langle A \mid \mathrm{Im}(G) \rangle = X = R^n.$$

Note that

$$\begin{aligned} \langle A \mid \mathrm{Im}(G) \rangle &= \langle A \mid \mathrm{Im}(G_1 + BV_u D^T \mid BV_u) \rangle \\ &= \langle A \mid \mathrm{Im}(G_1) \rangle \cup \langle A \mid \mathrm{Im}(B) \rangle. \end{aligned}$$

□

4. Canonical Form, Parametrization, and Identifiability

In this section, a subclass of Gaussian stochastic control systems is defined that is almost a canonical form. In addition a parametrization of this class is defined and an equivalent condition presented for the identifiability of this parametrization.

Consider a stationary Gaussian process taking values in R^{p+m} with zero mean value function. Denote by $W \colon T \to R^{(p+m)\times(p+m)}$ its covariance function. It follows from Theorem 3.1 that this process has a stochastic realization iff certain conditions are satisfied, see Theorem 3.1. Denote the class of covariance functions by $\mathbf{W}^{p+m}$ and by $W_S(p, n, m)$ the subclass of covariance functions that satisfy the condition of the theorem. Note that the minimal dimension of the state space n can be determined from the covariance function. The parameters of a Gaussian stochastic control system are determined by

$$gsp = \{A, B, C, D, M, N, V\} \in GSCSP(p, n, m, r).$$

Define the map

$$f\colon\ GSCSP(p, n, m, r) \longrightarrow W_S(p, n, m), \tag{37}$$

$$f(A, B, C, D, M, N, V) = \{W(t),\ t \in T\}, \tag{38}$$

$$W(t) = \begin{cases} \begin{pmatrix} CA^{t-1}G & CA^{t-1}BV_u \\ 0 & 0 \end{pmatrix}, & t > 0, \\ \begin{pmatrix} CQC^T + NVN^T + DV_uD^T & DV_u \\ V_uD^T & V_u \end{pmatrix}, & t = 0, \end{cases} \tag{39}$$

see Proposition 3.6, that associates gsp to the covariance function.

From Theorem 3.1 follows that if a covariance function admits a stochastic realization in the class of Gaussian stochastic control systems, that it may then admit many such realizations. First attention is restricted to a minimal realization. Even with this restriction, a stochastic realization is not unique. Therefore the map f is not injective. What is needed is a way to parametrize the set $W_S(p, n, m)$ by a subset $G_{CF} \subset GSCSP(p, n, m, r)$ for some $r \in Z_+$ such that the map $h\colon G_{CF} \rightarrow W_S(p, n, m)$ is injective and preferably continuous.

The problem of a canonical form for the class of Gaussian stochastic control systems is now motivated. Below definitions of a parametrization and a canonical form are stated.

4.1. PARAMETRIZATION AND CANONICAL FORM

DEFINITION 4.1. Let Y be a set. A *parametrization* of the set Y is a pair (X, f) with X a set and $f\colon X \rightarrow Y$ a map such that f is surjective. It is said to be *injective* if f is injective. It is said to *continuous* if X and Y are equipped with a topology and f is a continuous map with respect to the topology. A parametrization is said to be *identifiable* if it is injective.

The surjectiveness in Definition 4.1 is imposed because (X, f) must describe or parametrize all elements of Y. An example of a set to be parametrized is $Y = W_S(p, n, m)$ with parametrization (X, f) in which $X = GSCSP(p, n, m, r)$ with $r \in Z_+$, $r \geqslant p$, and f as defined in (39). Because by Theorem 3.1 any $W \in W_S(p, n, m)$ admits a minimal realization, f is surjective.

If a parametrization (X, f) of Y is not injective how can it then be transformed into an injective one? Define the relation $\sim$ on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. It follows directly from the definition of this relation that it is an equivalence relation. In Gaussian stochastic systems, an arbitrary stochastic realization and its Kalman realization are said to be equivalent if they correspond to the same covariance function.

DEFINITION 4.2. Let Y be a set and (X, f) be a parametrization of Y. Consider the above defined equivalence relation $\sim$ on X induced by f. A *canonical form* for (X, Y, f) is a triple (Z, g, h) consisting of a set Z and maps g: $X \to Z$, h: $Z \to Y$ such that

- for all $x \in X$ $f(x) = h(g(x))$, or, equivalently, $f = h \circ g$;
- the function g is surjective;
- the function h is injective.

THEOREM 4.3. *Let Y be a set and (X, f) be a parametrization of Y.*

(a) *There exists a canonical form (Z, g, h) of (X, Y, f).*
(b) *Suppose there exists two canonical forms (Z_1, g_1, h_1) and (Z_2, g_2, h_2) of (X, Y, f). Then there exists a bijection b: $Z_1 \to Z_2$ such that $g_2 = b \circ g_1$ and $h_1 = h_2 \circ b$.*

Proof. Not presented here. □

4.2. CANONICAL FORM FOR STOCHASTIC REALIZATIONS OF GAUSSIAN STOCHASTIC CONTROL SYSTEMS

To construct a canonical form for $(GSCSP(p, n, m, r), W_S(p, n, m), f)$ one has to specify a triple (Z, g, h). The canonical form for this set is not yet fully developed. Instead a set Z and maps g, h will be specified with the remaining equivalence in h.

DEFINITION 4.4. A Gaussian stochastic control system with as input process a Gaussian white noise process is said to be a *Kalman realization* of its observation process if its representation is of the form

$$x(t+1) = Ax(t) + \begin{pmatrix} K & B \end{pmatrix} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}, \tag{40}$$

$$z(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} I & D \\ 0 & I \end{pmatrix} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}, \tag{41}$$

with w, u independent Gaussian white noise processes with $w(t) \in G(0, V_w)$, $V_w = V_w^T > 0$, $u(t) \in G(0, V_u)$, $V_u = V_u^T > 0$, and

(1) $\mathrm{sp}(A) \subset C^-$;
(2) $\mathrm{sp}(A - KC) \subset C^-$;
(3) $(A, [KV_w \mid BV_u])$ is a reachable pair;
(4) (A, C) is an observable pair;
(5) $(A, [G_1 V_w \mid BV_u])$ is a reachable pair, where

$$G_1 = AQC^T + KV_w, \tag{42}$$

$$Q = AQA^T + KV_w K^T. \tag{43}$$

Denote the set of parameters of a Kalman realization by

$$GSCSP_{\text{KR}}(p,n,m,p) = \left\{ \begin{array}{c} (A,B,C,D,K,I,V_w,V_u) \\ \in R^{n\times n} \times R^{n\times m} \times R^{p\times n} \times R^{p\times m} \\ \times R^{n\times p} \times R^{p\times p} \times R^{p\times p} \times R^{m\times m} \\ \mid \text{satisfying (1) through (5) above} \end{array} \right\}. \tag{44}$$

Because of lack of space no realization algorithm is presented for the Kalman realization of an observed process.

THEOREM 4.5. *Consider a Gaussian stochastic control system that satisfies* $\text{sp}(A) \subset C^-$ *and with a Gaussian white noise input process.*

(a) *There exists a Gaussian stochastic control system that is a Kalman realization of the observation process associated with the given system.*
(b) *Any two Kalman realizations of the same observation process are related by a state space transformation, or, if the Kalman realizations are represented by*

$$\{(A_1, B_1, C_1, D_1, K_1, I, V_{1w}, V_{1u}) \in GSCSP_{\text{KR}}(p, n_1, m, p)\},$$

$$\{(A_2, B_2, C_2, D_2, K_2, I, V_{2w}, V_{2u}) \in GSCSP_{\text{KR}}(p, n_2, m, p)\}$$

then $n = n_1 = n_2$ *and there exists a nonsingular* $S \in R^{n\times n}$ *such that*

$$\begin{array}{llll} A_1 = SA_2S^{-1}, & B_1 = SB_2, & C_1 = C_2S^{-1}, & D_1 = D_2, \\ K_1 = SK_2, & V_{1w} = V_{2w}, & V_{1u} = V_{2u}. & \end{array} \tag{45}$$

Proof. (a) Given a Gaussian stochastic control system one may construct its observation process.

From Theorem 3.1 follows that there exists a minimal stochastic realization.

From [6, 11] it follows that there exists a minimal stochastic realization with $\text{sp}(A) \subset C^-$ and $\text{sp}(A - KC) \subset C^-$.

(b) This follows from the corresponding result for Gaussian stochastic systems. □

In the case of a single output system, with $p = 1$, one may take the observable canonical form for the pair (A, C). This subclass is then truly a canonical form.

In [14, pp. 101–106] identifiability of Gaussian stochastic systems is discussed. The representation used is an ARMAX representation. The condition imposed in that book, see [14, p. 103], includes that the ARMAX representation is a predictor. This condition is equivalent to condition (2) of Definition 4.4, or $\text{sp}(A - KC) \subset C^-$.

In [8, pp. 44–49] a canonical form is derived only for the case in which one starts with the spectral factor.

It is conjectured that there does not exist a continuous canonical form for $W_S(p, n, m)$ if $p > 1$ and $m > 1$. This conjecture is inspired by a corresponding result for finite-dimensional linear systems, see [9].

4.3. STRUCTURED GAUSSIAN STOCHASTIC CONTROL SYSTEMS

Structural identifiability of deterministic finite-dimensional linear systems is discussed in many papers, see for an introduction to the literature [16].

As argued before, the subset of Gaussian stochastic control systems that are Kalman realizations may be taken as a starting point for a canonical form. For structured Gaussian stochastic control systems attention is therefore restricted to this class of realizations.

DEFINITION 4.6. A *structured Kalman realization* of a Gaussian stochastic control system with as input process Gaussian white noise is a set $P \subset R^s$ for some $s \in Z_+$, and maps

$$A\colon P \longrightarrow R^{n\times n}, \qquad B\colon P \longrightarrow R^{n\times m},$$
$$C\colon P \longrightarrow R^{p\times n}, \qquad D\colon P \longrightarrow R^{p\times m},$$
$$K\colon P \longrightarrow R^{n\times p}, \qquad V_w\colon P \longrightarrow R^{p\times p}, \qquad V_u\colon P \longrightarrow R^{m\times m},$$

such that for any $q \in P$

$$\{A(q), B(q), C(q), D(q), K(q), I, V_w(q), V_u(q)\} \in GSCSP_{\mathrm{KR}}(p,n,m,p)$$

is the parameter of a Kalman realization of a Gaussian stochastic control system. This system is represented by

$$x(t+1) = A(q)x(t) + \begin{pmatrix} K(q) & B(q) \end{pmatrix}\begin{pmatrix} w(t) \\ u(t) \end{pmatrix}, \tag{46}$$

$$z(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} C(q) \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} I & D(q) \\ 0 & I \end{pmatrix}\begin{pmatrix} w(t) \\ u(t) \end{pmatrix}, \tag{47}$$

where $w(t) \in G(0, V_w(q))$ and $u(t) \in G(0, V_u(q))$. Denote by

$$GSCSP_{\mathrm{SKR}}(p,n,m,p) = \{(A(q),\ldots,V_u(q)) \in GSCSP_{\mathrm{KR}} \mid q \in P\}$$

the set of possible parameters of this structured system and by f_1 the map

$$f_1\colon P \longrightarrow GSCSP_{\mathrm{SKR}}(p,n,m,p), \qquad q \longmapsto (A(q),\ldots,V_u(q)).$$

Note that (P, f_1) is a parametrization of the set $GSCSP_{\mathrm{SKR}}(p,n,m,p)$. Of interest is the parametrization of the set of covariance functions associated with the observation process of such a system, say $W_{\mathrm{Str}}(p,n,m)$. Let

$$f \circ f_1\colon P \longrightarrow W_{\mathrm{Str}}(p,n,m) \tag{48}$$

be the composition of the maps f_1 with f, where f is as defined in (37). Then $(P, f \circ f_1)$ is a parametrization of $W_{\text{Str}}(p, n, m)$. Is this parametrization identifiable?

In the following, terminology of algebraic geometry is used, see [3] and [10, Chapter X, Section 3]. A property is said to be *generic* on $P \subset R^n$ if it holds for all $p \in P$ outside an algebraic set. An *algebraic set* is defined by a finite set of polynomials according to

$$\{q \in P \mid g_1(q) = 0, \ldots, g_k(q) = 0\},$$

where $g_i\colon P \to R$, $i = 1, \ldots, k$ are polynomials.

DEFINITION 4.7. Consider a structured Gaussian stochastic control system with as input process a Gaussian white noise process that is observed. Consider the parametrization (P, f_1) of $GSCSP_{\text{SKR}}$ as defined in Definition 4.6. This parametrization is said to be *structurally identifiable* (from the covariance function) if generically the parametrization $(P, f \circ f_1)$ of the associated set of covariance functions is identifiable. Thus, it is structurally identifiable if the map $f \circ f_1\colon P \to W_{\text{Str}}$ is injective outside an algebraic set.

THEOREM 4.8. *Consider a structured Gaussian stochastic control system with representation*

$$x(t+1) = A(q)x(t) + \begin{pmatrix} K(q) & B(q) \end{pmatrix} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}, \tag{49}$$

$$z(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} C(q) \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} I & D(q) \\ 0 & I \end{pmatrix} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}, \tag{50}$$

with

$$W(t) \in G(0, V_w(q)), \qquad V_w(q) = V_w(q)^T > 0,$$

$$u(t) \in G(0, V_u(q)), \qquad V_u(q) = V_u(q)^T > 0,$$

and parametrization (P, f_1) as defined in Definition 4.6. *This representation need not satisfy the conditions of a Kalman realization.*

(a) *The representation* (49), (50) *is a structurally minimal stochastic realization of its observation process iff for all $q \in P$ outside an algebraic set*

(1) $\text{sp}(A(q)) \subset C^-$;
(2) $\text{sp}(A(q) - K(q)C(q)) \subset C^-$;
(3) $(A(q), [K(q)V_w(q) \mid B(q)V_u(q)])$ *is a reachable pair;*
(4) $(A(q), [G_1(q) \mid B(q)V_u(q)])$ *is a reachable pair, where $G_1(q)$ is as G_1 in* (42);
(5) $(A(q), C(q))$ *is an observable pair.*

(b) *Assume that the system is a structurally minimal stochastic realization. The parametrization (P, f_1) of $GSCSP_{SKR}$ is structurally identifiable iff for all $q_1, q_2 \in P$ outside an algebraic set and $S \in R^{n \times n}$ nonsingular*

$$A(q_1) = SA(q_2)S^{-1}, \qquad B(q_1) = SB(q_2), \\ C(q_1) = C(q_2)S^{-1}, \qquad D(q_1) = D(q_2), \tag{51}$$

$$K(q_1) = SK(q_2), \qquad V_w(q_1) = V_w(q_2), \qquad V_u(q_1) = V_u(q_2), \tag{52}$$

imply that $q_1 = q_2$.

Proof. This follows directly from Theorem 4.5. □

5. Conclusion

The weak stochastic realization problem for a Gaussian stochastic control system has been formulated and solved. An equivalent condition for the existence of a stochastic realization has been presented and an algorithm provided. The subclass of Kalman realizations has been defined and it has been established that this class is almost a canonical form for this problem. The remaining invariance has been exhibited. Identifiability conditions for a structured Gaussian stochastic control system have been presented.

Extensions and open questions of the problem considered in this paper there are many. The case in which the input process is not Gaussian white noise but a colored Gaussian process can be treated along the same lines. More interesting is a more explicit canonical form that avoids the difficult to verify stochastic reconstructibility condition (5) of Definition 4.4 or condition (4) of Theorem 4.8.a.

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